

D6 branes and M–theory geometrical transitions from gauged supergravity

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We study the supergravity duals of supersymmetric theories arising in the world–volume of D6 branes wrapping holomorphic two–cycles and special Lagrangian three–cycles within the framework of eight dimensional gauged supergravity. When uplifted to 11d, our solutions represent M–theory on the background of, respectively, the small resolution of the conifold and a manifold with G_2 holonomy. We further discuss on the flop and other possible geometrical transitions and its implications.

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1. Introduction

The world-volume low-energy dynamics of D-branes in certain curved backgrounds defines a topologically twisted supersymmetric field theory [1]. The twisting is necessary to allow for the world-volume of the brane to support covariantly constant spinors (this is reminiscent of a similar phenomenon in lower dimensional supergravities [2]). If the D-brane is wrapping a nontrivial cycle, and we take its size to zero, the infrared dynamics of the system is described by a lower dimensional field theory with either *ordinary* or *twisted* (depending on the higher dimensional twisting being respectively *partial* or *full*) *reduced* supersymmetry [3]. The amount of supersymmetry preserved has to do with the way in which the cycle is embedded in a higher dimensional space. If the number of branes is taken to be large, this sort of systems provide a supergravity dual description of $\mathcal{N} = 1$ or $\mathcal{N} = 2$ supersymmetric field theories [4][5][6][7][8].

In this paper we will consider D6 brane configurations that reduce, at low energies, to theories with four and eight supercharges in four and five dimensions. The D6 brane system is best described in the infrared by means of $\mathcal{N} = 2$ seven dimensional super Yang-Mills theory [9]. So, for example, wrapping these branes on S^3 would imply, after appropriate twisting, breaking one quarter of the supercharges, the theory reducing to pure $\mathcal{N} = 1$ four dimensional super Yang-Mills in the infrared. The above referred twisting corresponds to S^3 being a special Lagrangian submanifold of a Calabi-Yau threefold, namely, the deformed conifold T^*S^3 . On the other hand, if the D6 branes wrap a holomorphic S^2 in the cotangent bundle of S^2 , T^*S^2 , the infrared dynamics will be governed by five dimensional $\mathcal{N} = 2$ super Yang-Mills theory.

It was recently proposed that the configuration of D6 branes wrapping an S^3 in T^*S^3 is dual, through a conifold transition, to a type IIA geometry where the D6 branes have disappeared, being replaced by RR fluxes on the blowup S^2 [10]. Conversely, there is a mirrored type IIB version of this phenomenon with D5 branes wrapping the S^2 becoming RR fluxes on the S^3 . It was almost immediately realized that this duality can be better viewed in M-theory on G_2 holonomy manifolds [11], where it corresponds to a flop transition [12]. It is natural to analyze these configurations in 11d for the fact that uplifted D6 branes become purely gravitational. Besides, the D6 branes are strongly coupled in the ultraviolet and the would be decoupling limit has to be addressed in eleven dimensions. In particular, the 11d supergravity solution is trustable for any number of branes. Another difference with other D-branes is given by the fact that massive geodesics can escape to infinity signaling the non decoupling of gravity [9].

It is our purpose in this paper to study this sort of solutions under the light of lower dimensional gauged supergravity. This is the natural framework to perform twisting. The solutions emerging from these theories correspond to near horizon D-brane solutions thus giving directly the gravity duals of gauge theories living on the world-volume of the brane. Since we will work with D6 branes, the twisting would require to impose boundary conditions on eight dimensional gravitational, gauge and scalar fields so, following the methods introduced in [4], the natural set up for this problem is eight dimensional gauged supergravity. In particular, we will work within the framework of *maximal 8d* gauged supergravity [13] so as to have enough room for different twistings. The virtue of gauged supergravities in this respect is that they provide quite cleanly the gauge field modes that undertake the partial twisting.

Uplifting to eleven dimensions will leave us with M -theory on Ricci flat backgrounds corresponding to the small resolution of the conifold and a G_2 holonomy manifold. Both manifolds eventually develop singularities where transitions to a different manifold might be possible. In the latter case, for example, the geometrical transition correspond to the above referred flop between two three-spheres that, at the singular point, constitute the base of a cone [12]. Instead, in the former case, we found that there is no transition, and the theory in the ultraviolet falls into the singularity. The reason for the absence of a geometrical transition can be attributed, as we will discuss, to the non existence of a θ angle in five dimensional theories. This suggest that a duality between large N five dimensional $\mathcal{N} = 2$ super Yang-Mills theory and superstrings propagating in a K3 manifold with fluxes turned on, in the spirit of [10], does not take place.

The plan of the paper is as follows. In Section 2 we review maximal gauged supergravity in eight dimensions and prepare the set up for the search of solutions. Section 3 is devoted to the case of D6 branes wrapping special Lagrangian 3-cycles. We first construct solutions of 8d gauged supergravity that are subsequently uplifted to 11d. The resulting geometry is that of a G_2 holonomy manifold recently studied in [12]. In section 4 we consider the case of D6 branes on holomorphic 2-cycles in a deformed A_1 singularity of $K3$, namely T^*S^2 . When uplifted to 11d our solution is the small resolution of the conifold $\mathcal{O}(-1) + \mathcal{O}(-1) \rightarrow \mathbb{P}^1$. We discuss on the obstructions to the geometrical transitions appearing in this case and their relation to generic aspects of five dimensional gauge theories. We conclude in section 5 with a discussion of our results, and an outlook of avenues for further research.

Note Added: While the final version of this paper was being typewritten, some results that overlap part of ours were reported by Jaume Gomis [14].

2. Review of $d = 8$ gauged supergravity

Maximal gauged supergravity in eight dimensions was originally constructed by Salam and Sezgin [13]. It arises from dimensional reduction of 11d supergravity on a $SU(2)$ group manifold [15]. The field content of this theory consists of the metric $g_{\mu\nu}$, a dilatonic scalar Φ , five scalars given by a unimodular 3×3 matrix L_α^i in the coset $SL(3, \mathbb{R})/SO(3)$, a seventh scalar B , a three-form $B_{(3)}$, three two-forms $B_{(2)}^i$, three vector fields $B_{(1)}^i$ and a $SU(2)$ gauge potential A_μ^i , as well as the pseudo Majorana spinors ψ_μ and χ_i . In this paper we are going to restrict ourselves to a sector of the theory with vanishing B -fields. This amounts to pure gravitational solutions of the 11d system. The bosonic dynamics in this sector is governed by the Lagrangian

$$e^{-1}\mathcal{L} = \frac{1}{4}R - \frac{1}{4}e^{2\Phi}F_{\mu\nu}^i F^{\mu\nu i} - \frac{1}{4}P_{\mu ij}P^{\mu ij} - \frac{1}{2}(\partial_\mu \Phi)^2 - \frac{g^2}{16}e^{-2\Phi}(T_{ij}T^{ij} - \frac{1}{2}T^2) , \quad (2.1)$$

where e is the determinant of the *achtbein* e_μ^a , $F_{\mu\nu}^i$ is the Yang–Mills field strength and $P_{\mu ij}$ is a symmetric and traceless quantity defined by

$$P_{\mu ij} + Q_{\mu ij} \equiv L_i^\alpha (\partial_\mu \delta_\alpha^\beta - g\epsilon_{\alpha\beta\gamma}A_\mu^\gamma)L_{\beta j} , \quad (2.2)$$

$Q_{\mu ij}$ being the antisymmetric counterpart. We have set $\kappa = 1$. As usual, greek indices are curved (α, β, \dots are in the group manifold ¹ and μ, ν, \dots label space–time coordinates) while latin ones are flat. Notice that, for example, $A_\mu^\gamma = L_i^\gamma A_\mu^i$, as well as $F_{\mu\nu}^\alpha = L_i^\alpha F_{\mu\nu}^i$. Finally, the potential energy corresponding to the scalar fields is governed by the so-called T -tensor,

$$T^{ij} = L_\alpha^i L_\beta^j \delta^{\alpha\beta} , \quad (2.3)$$

and $T = \delta_{ij}T^{ij}$. The equations of motion are

$$R_{\mu\nu} = P_{\mu ij}P_\nu^{ij} + 2\partial_\mu \Phi \partial_\nu \Phi + 2e^{2\Phi}F_{\mu\gamma}^i F_\nu^{\gamma i} - \frac{1}{3}g_{\mu\nu}\nabla^2 \Phi , \quad (2.4)$$

$$\nabla_\mu (e^{2\Phi}F^{\mu\nu i}) = -e^{2\Phi}P_\mu^{ij}F_j^{\mu\nu} - gg^{\nu\gamma}\epsilon^{ijk}P_{\gamma jl}T_k^l , \quad (2.5)$$

$$\nabla_\mu P^{\mu ij} = -\frac{2}{3}\delta^{ij}\nabla^2 \Phi + e^{2\Phi}F_{\mu\nu}^i F^{\mu\nu j} + \frac{g^2}{2}e^{-2\Phi}\Theta^{ij} , \quad (2.6)$$

¹ While working in eight dimensions, these indices describe a flat space. The dependence on the coordinates of the group manifold have been factored out, and it will only reappear when uplifting to 11d is performed.

where Θ^{ij} is short for

$$\Theta^{ij} \equiv T^i_k T^{jk} - \frac{1}{2} T T^{ij} - \frac{1}{2} \delta^{ij} (T_{kl} T^{kl} - \frac{1}{2} T^2) . \quad (2.7)$$

Notice that the dilaton equation is obtained from (2.6) by tracing over the latin indices.

The supersymmetry transformations for the fermions are given by

$$\delta\psi_\gamma = \mathcal{D}_\gamma \epsilon + \frac{1}{24} e^\Phi F_{\mu\nu}^i \Gamma_i (\Gamma_\gamma^{\mu\nu} - 10 \delta_\gamma^\mu \Gamma^\nu) \epsilon - \frac{g}{288} e^{-\Phi} \epsilon_{ijk} \Gamma^{ijk} \Gamma_\gamma T \epsilon , \quad (2.8)$$

$$\delta\chi_i = \frac{1}{2} (P_{\mu ij} + \frac{2}{3} \delta_{ij} \partial_\mu \Phi) \Gamma^j \Gamma^\mu \epsilon - \frac{1}{4} e^\Phi F_{\mu\nu i} \Gamma^{\mu\nu} \epsilon - \frac{g}{8} e^{-\Phi} (T_{ij} - \frac{1}{2} \delta_{ij} T) \epsilon^{jkl} \Gamma_{kl} \epsilon , \quad (2.9)$$

where the covariant derivative is

$$\mathcal{D}_\mu \epsilon = (\partial_\mu + \frac{1}{4} \omega_\mu^{ab} \Gamma_{ab} + \frac{1}{4} Q_{\mu ij} \Gamma^{ij}) \epsilon . \quad (2.10)$$

It is useful for later purposes to work alternatively with spinors of 32 components or doublets of sixteen components. Then, we will use the following representation for the Clifford algebra

$$\Gamma^a = \gamma^a \times \mathbb{I} \quad \Gamma^i = \gamma_9 \times \sigma^i , \quad (2.11)$$

where γ^a are eight dimensional gamma matrices (a being a flat index), $\gamma_9 = i\gamma^0\gamma^1\cdots\gamma^7$, with $\gamma_9^2 = 1$, and σ^i are the Pauli matrices corresponding to the R -symmetry group. It will be finally convenient to introduce $\Gamma_9 \equiv \frac{1}{6i} \epsilon_{ijk} \Gamma^{ijk} = \gamma_9 \times \mathbb{I}$.

In the following we will consider supergravity duals of gauge theories in four and five dimensions with four and eight supercharges respectively. Our procedure is based on taking the low energy limit for a D6 brane wrapped on three and two supersymmetric cycles in Calabi–Yau and $K3$ manifolds. The structure group of the normal bundle of these cycles is, respectively, $SO(3)$ and $SO(2)$, thus Salam–Sezgin theory has enough room for their twisting. When the energies are low enough, the cycle decouples and we remain with a theory that has less dimensions and less supersymmetries than the original one.

Since we will work with D6 branes, it seems natural to consider seven dimensional boundary conditions for gauge and scalar fields, so, the natural set up for this problem is eight dimensional gauged supergravity. We can see that the vacuum supersymmetric solution of this theory is given by

$$ds_8^2 = e^{\frac{2}{3}\phi} dx_{1,6}^2 + dr^2 , \quad (2.12)$$

$$e^{\phi-\phi_0} = r , \quad (2.13)$$

where $\phi_0 = \log(\frac{3g}{8})$. When uplifted to eleven dimensions by means of the prescription given in Ref.[13], after appropriate coordinate rescaling, the higher dimensional configuration is

$$ds^2 = dx_{1,6}^2 + N(d\rho^2 + \rho^2 d\Omega_3) . \quad (2.14)$$

After modding out the outer three-sphere by \mathbb{Z}_N , we get an ALE space with an A_{N-1} singularity in coincidence with the uplifting of the near horizon solution corresponding to D6 branes in type IIA [9].

3. D6 branes on the deformed conifold

In this section we will obtain the gravity dual of $\mathcal{N} = 1$ super Yang–Mills theory in four dimensions, arising in the low–energy dynamics of D6 branes wrapped on S^3 in T^*S^3 , starting from eight dimensional gauged supergravity. Let us start with an ansatz for the metric that describes such deformation of the world–volume of the D6 brane

$$ds^2 = e^{2f} dx_{1,3}^2 + e^{2h} d\Omega_3 + dr^2 , \quad (3.1)$$

where $d\Omega_3$ is the metric of the unit three–sphere. As explained in the introduction, wrapping the branes on a curved cycle implies that the theory has to be twisted on the curved part.

The fields on the D6 branes transform under $SO(1,6) \times SO(3)_R$ as **(8,2)** for the fermions and **(1,3)** for the scalars, while the gauge field is a singlet under R –symmetry. When we wrap the D6 branes on a three–cycle, the symmetry group splits as $SO(1,3) \times SO(3) \times SO(3)_R$, and we shall construct a diagonal $SO(3)_D$ from the $SO(3)$ of the cycle and the one of the R –symmetry (in other words, we mix the spin connection with the gauge connection, as explained above). It can be easily seen that the effect of the twisting is to preserve the vector fields but transforms the scalars in one forms on the curved surface, so we are left with a theory with no scalars fields in the infrared; besides four supercharges are preserved.

We will describe the S^3 as a $SU(2)$ group manifold by means of the left invariant forms w^i ,

$$\begin{aligned} w^1 &= \cos \phi \, d\theta + \sin \theta \sin \phi \, d\psi , \\ w^2 &= \sin \phi \, d\theta - \sin \theta \cos \phi \, d\psi , \\ w^3 &= d\phi + \cos \theta \, d\psi , \end{aligned} \quad (3.2)$$

satisfying

$$dw^i = \frac{1}{2}\epsilon^{ijk}w^jw^k , \quad (3.3)$$

in terms of which the metric of the unit sphere simply reads

$$d\Omega_3 = \frac{1}{4} \sum_{i=1}^3 (w^i)^2 . \quad (3.4)$$

The twisting is achieved by turning on a non-Abelian $SO(3)$ gauge field given by the left invariant form of the three sphere,

$$A^i = -(2g)^{-1}w^i , \quad (3.5)$$

whose field strength

$$F^i = -(8g)^{-1}\epsilon^{ijk}w^jw^k , \quad (3.6)$$

trivially obeys the corresponding equation of motion. This corresponds to a complete identification of the spin connection with the R-symmetry. In such case it is possible to get rid of the scalars L_α^i ,

$$L_\alpha^i = \delta_\alpha^i \quad \Rightarrow \quad P_{ij} = 0 \quad , \quad Q_{ij} = -g\epsilon_{ijk}A^k . \quad (3.7)$$

The T -tensor is drastically simplified to $T_{ij} = \delta_{ij}$, $T = 3$, and $\Theta_{ij} = \frac{1}{4}\delta_{ij}$. Supersymmetric configurations require the following projections in the parameter ϵ :

$$\gamma_{\underline{t}} \epsilon = -i\gamma_9 \epsilon \quad \gamma_{ab} \epsilon = -\sigma^{ab} \epsilon , \quad (3.8)$$

where $a, b = \underline{t}, \underline{\phi}, \underline{\psi} \equiv 1, 2, 3$ are the directions along the three-sphere. These projection leave unbroken 1/8 of the original supersymmetries, that is, four supercharges. The first order BPS equations are,

$$f' = \frac{1}{3}\Phi' = -\frac{1}{2g}e^{\Phi-2h} + \frac{g}{8}e^{-\Phi} , \quad (3.9)$$

$$h' = \frac{3}{2g}e^{\Phi-2h} + \frac{g}{8}e^{-\Phi} . \quad (3.10)$$

By simple inspection, we can quickly find a solution

$$e^{2\Phi} = \frac{g^2}{16}r^2 , \quad e^{2h} = \frac{3}{4}r^2 . \quad (3.11)$$

Notice that the relation $\Phi' = 3f'$ is forced from the Ricci flatness of the corresponding eleven dimensional solution. When uplifted to 11d, we obtain a Ricci flat solution of the form $M_4 \times Y_7$ where Y_7 is a cone whose base X_6 is an Einstein manifold with the topology of $S^3 \times \tilde{S}^3$,

$$ds^2 = dx_{1,3}^2 + dr^2 + \frac{r^2}{9} [(w^a)^2 + (\tilde{w}^a)^2 - w^a \tilde{w}^a] , \quad (3.12)$$

\tilde{w}^a being the left invariant one forms associated with \tilde{S}^3 . This metric coincides with the asymptotic at large r of the M -theory solution on a G_2 holonomy manifold studied in Ref.[12], and we note that the solution is singular in the infrared. It is natural to try to obtain a solution where the singularity is absent. In this system we do not have further degrees of freedom to turn on, that could occasionally solve the singularity; this means that there must be other solutions to the BPS equations (3.9)(3.10), such that, when uplifted to eleven dimensions, do not give place to singularities in the infrared.

We can define indeed a pair of functions, $u \equiv h + \Phi$ and $v \equiv 3h - \Phi$, the system simplifies to $e^u du = \frac{g^2}{12} e^v dv$, whose immediate solution is

$$e^u = \frac{g^2}{12} \left(e^v - \frac{a^3}{3^{\frac{3}{2}}} \right) , \quad (3.13)$$

a being a constant. There is an amusing change of variable

$$r(\rho) = \frac{(2g)^{\frac{1}{2}}}{3^{\frac{3}{2}}} \left(\rho^{\frac{3}{2}} {}_2F_1\left[-\frac{1}{2}; \frac{1}{4}, \frac{1}{2}; \frac{a^3}{\rho^3}\right] - a^{\frac{3}{2}} \frac{\sqrt{\pi}\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})} \right) , \quad (3.14)$$

where ${}_2F_1[a, b, c, z]$ is the hypergeometric function ²

$${}_2F_1[a, b, c, z] = \sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{(c)_m} \frac{z^m}{m!} , \quad (3.15)$$

with $(a)_n = \Gamma(a+n)/\Gamma(a)$ the Pochhammer symbol, that allows us to find a solution to the BPS equations of the form

$$e^{2\Phi} = \frac{g^3}{216} \rho^3 \left(1 - \frac{a^3}{\rho^3} \right)^{\frac{3}{2}} \quad e^{2h} = \frac{g}{18} \rho^3 \left(1 - \frac{a^3}{\rho^3} \right)^{\frac{1}{2}} . \quad (3.16)$$

² Notice that in our case, for $\rho \geq a$, it has a real variable $z \leq 1$ such that the change of variables has not branch cut discontinuity. The subtracted constant in (3.14) just amounts to $r(a) = 0$.